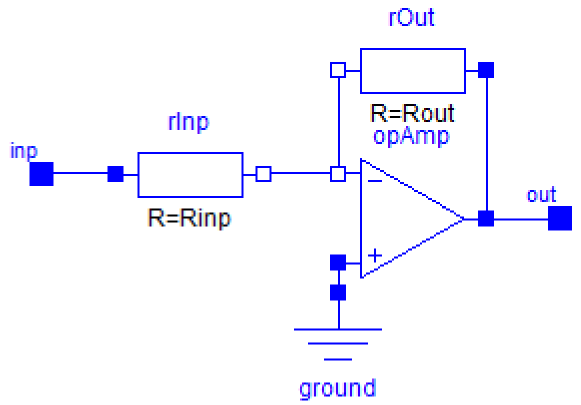
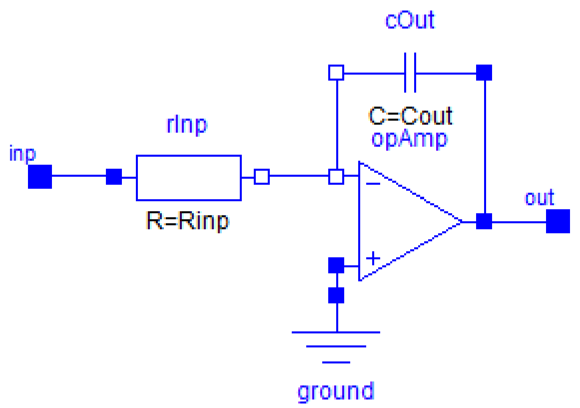
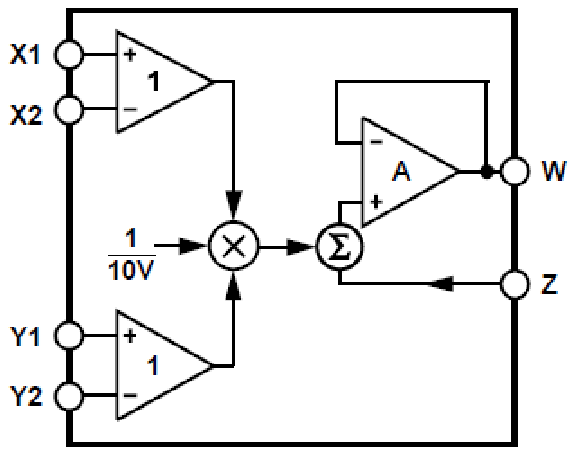
Comparison of chaotic circuits

Lotka-Volterra	$\frac{dx_i}{dt} = r_i \cdot x_i \cdot \left(1 - \sum_{j=1}^{n} a_{ij} \cdot x_j\right)$
van der Pol Circuit	$\frac{dx}{d\tau} = y$ $y = \mu \cdot \left[1 - \frac{1}{3} \cdot x^2\right] \cdot x - z + \begin{cases} 0 \\ A \cdot cos(w \cdot \tau - \pi) \end{cases}$ $\frac{dz}{d\tau} = x$
Lorenz System	$\frac{dx}{dt} = \sigma \cdot (y - x)$ $\frac{dy}{dt} = x \cdot (\rho - z) - y$ $\frac{dz}{dt} = x \cdot y - \beta \cdot z$
Roessler System	$\frac{dx}{dt} = -y - z$ $\frac{dy}{dt} = x + a \cdot y$ $\frac{dz}{dt} = b + (x - c) \cdot z$
Chua's Circuit	nonlinear conductor (two NICs) with partly negative slope
Chaotic Diode Circuit	$\frac{dx}{dt} = -y$ $\frac{dy}{dt} = -z$ $\frac{dz}{dt} = -x + a \cdot (e^y - 1) - b \cdot z$
Chaotic Oscillator A simple chaotic oscillator for educational purposes	$\frac{dx}{dt} = y$ $\frac{dy}{dt} = a \cdot y - x - z$ $\frac{dz}{dt} = b + y - c \cdot (e^z - 1)$
Colpitts Oscillator	LC oscillator with transistor (orig. vacuum tube)
Shinriki Oscillator	two antiparallel Z-diodes and nonlinear conductor (NIC) with partly negative slope
Jerk Circuit	$\dddot{x} + G(\ddot{x}, \dot{x}, x) = 0$
Rikitake System	Two coupled disc dynamos.
Chua's Circuit with Memristor	Replace Chua's diode with a special memristor
RLD - Resonator	Series resonant circuit with diode in parallel to the capacitor

OpAmp-Circuits

Algebraic-Differential Equation Systems can be simulated with analog computers
using operational amplifiers and analog multipliers.

Amplifier/Adder  	$\frac{inp.v}{R_{inp}} + \frac{out.v}{R_{out}} = 0$ $-out.v = k \cdot inp.v$ $k = \frac{R_{out}}{R_{inp}}$ Input resistance $= R_{inp}$ Output resistance $\rightarrow 0$ It is possible to add several inputs.
Integrator  	$\frac{inp.v}{R_{inp}} + C_{out} \cdot \frac{d\,out.v}{dt} = 0$ $-out.v = v_0 + \frac{1}{\tau} \cdot \int_0^t inp.v \cdot dt$ $\tau = R_{inp} \cdot C_{out}$ Input resistance $= R_{inp}$ Output resistance $\rightarrow 0$ It is possible to integrate the sum of several inputs.
Analog Multiplier  	Functional Block Diagram of AD633 Division by 10 V (scaling) inhibits overflow. Additional summing input Z is omitted. Negative inputs of X- and Y-amplifiers are connected to ground. Possible implementations: • Gilbert cell • $y = e^{ln(x_1)+ln(x_2)}$ • $y = \frac{(x_1+x_2)^2-(x_1-x_2)^2}{4}$

Lotka-Volterra

[Vano2006]

As a 2-dimensional predator-prey-model no chaos is reported:

$$\frac{dx}{dt} = r_x \cdot x - d_x \cdot x \cdot y$$
$$\frac{dy}{dt} = d_x \cdot x \cdot e_y \cdot y - d_y \cdot y$$

We might interpret x as number of hares (prey) and y as foxed (predator).
r_x is the reproduction rate of hares, d_x the deathrate of hares due to foxes.

e_y is the efficiency in growing foxes from hares, d_y the (natural) deathrate of foxes.

A n-dimensional Lotka-Volterra model is defined for $2 \leq i \leq n$ (n designates the number of species):

$$\frac{dx_i}{dt} = r_i \cdot x_i \cdot \left(1 - \sum_{j=1}^{n} a_{ij} \cdot x_j\right)$$

The vector $[r]$ describes the reproduction rates of the species whereas the quadratic matrix $[a]$ describes the competition between species.

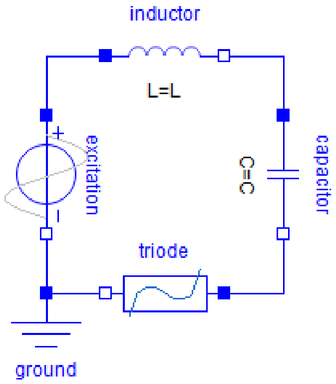
The above mentioned predator-prey case with 2 species can be expressed as:

$$[r] = \begin{bmatrix} r_x \\ -d_y \end{bmatrix}$$
$$[a] = \begin{bmatrix} 0 & \frac{d_x}{r_x} \\ \frac{d_x \cdot e_y}{d_y} & 0 \end{bmatrix}$$

van der Pol Circuit

[Buscarino2014], [VanDerPol1927]

Balthasar van der Pol reported 1927 strange phenomena about oscillations in a series resonance circuit containing a vacuum electron triode. Due to the nonlinear characteristic of the triode the autonomous circuit is able to maintain periodic oscillations, and with harmonic excitation it is able to produce chaos.



$$i = C \cdot \frac{dv_C}{dt} \rightarrow v_C = v_{C0} + \frac{1}{C} \cdot \int_{t_0}^{t} i \cdot dt'$$

$$L \cdot \frac{di}{dt} + \tilde{R}(i) \cdot i + v_C = \begin{cases} 0 \\ \hat{V} \cdot cos(\omega \cdot t - \pi) \end{cases}$$

$$\tilde{R}(i) = -R_0 \cdot \left[1 - \frac{1}{3} \cdot \left(\frac{i}{I_0}\right)^2\right]$$

These are the equations of the physical model.
Note the phase shift of the excitation!

$$L \cdot \frac{d^2 i}{dt^2} - R_0 \cdot \left[1 - \left(\frac{i}{I_0}\right)^2\right] \cdot \frac{di}{dt} + \frac{1}{C} \cdot i = \begin{cases} 0 \\ \hat{V} \cdot \omega \cdot sin(\omega \cdot t) \end{cases}$$

$$\omega_0 = \frac{1}{\sqrt{L \cdot C}}$$

$$\mu = R_0 \cdot \sqrt{\frac{C}{L}}$$

$$A = \frac{\omega_0 \cdot C \cdot \hat{V}}{I_0}$$

$$w = \frac{\omega}{\omega_0}$$

$$\tau = \omega_0 \cdot t$$

$$x = \frac{i}{I_0}$$

$$y = \frac{dx}{d\tau} = \frac{1}{\omega_0 \cdot I_0} \cdot \frac{di}{dt}$$

$$\frac{I_0}{\omega_0 \cdot C} \cdot z = v_C = v_{C0} + \frac{1}{C} \cdot \int_{t_0}^{t} i \cdot dt'$$

$$z = z_0 + \int_{\tau_0}^{\tau} x \cdot d\tau'$$

$\frac{d^2 x}{d\tau^2} - \mu \cdot [1 - x^2] \cdot \frac{dx}{d\tau} + x = \begin{cases} 0 \\ A \cdot w \cdot sin(w \cdot \tau) \end{cases}$ $\frac{dx}{d\tau} = y$ $\frac{dy}{d\tau} = \mu \cdot [1 - x^2] \cdot y - x + \begin{cases} 0 \\ A \cdot w \cdot sin(w \cdot \tau) \end{cases}$ $\frac{dz}{d\tau} = x$	Alternative Formulation with 2 states: $\frac{dx}{d\tau} = y$ $y = \mu \cdot \left[1 - \frac{1}{3} \cdot x^2\right] \cdot x - z + \begin{cases} 0 \\ A \cdot cos(w \cdot \tau - \pi) \end{cases}$ $\frac{dz}{d\tau} = x$

Note:
Instead of using a series resonance circuit and deriving a scaled differential equation for the current, we could use an equivalent parallel resonance circuit and derive a scaled differential equation for the voltage.

$$v = L \cdot \frac{di_L}{dt} \rightarrow i_L = i_{L0} + \frac{1}{L} \cdot \int_{t_0}^{t} v \cdot dt'$$

$$i_L + \tilde{G}(v) \cdot v + C \cdot \frac{dv}{dt} = i_e$$

Initialization:
The physical model has 2 states: i and v_C. Current i acts as an initial value for the nonlinear resistor.
The analytic equations have 3 states: x, y and z.
The third state has been introduced artificially by first differentiating the voltage equation, generating an equation with second derivative of i. Splitting this equation into two first order differential equations, we generate i and $\frac{di}{dt}$ as states. Calculating capacitor voltage v, we get the third state.

For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. We chose natural eigen frequency ω_0 as time scale:

$$x' = \frac{x}{k_x}$$

$$y' = \frac{y}{k_y}$$

$$z' = \frac{z}{k_z}$$

$$t' = \frac{\tau}{\omega_0}$$

We also have to take into account that the analog multiplier divides by V_S to avoid overflow of the output. After that, none of the computing block should encounter an overflow.
This leads to the following set of equations:

$$\frac{1}{\omega_0} \cdot \frac{dx'}{dt'} = \frac{k_y}{k_x} \cdot y'$$

$$y' = \mu \cdot \frac{k_x \cdot V_s}{k_y} \cdot \left[1 - \frac{{k_x}^2 \cdot V_s}{3} \cdot \frac{{x'}^2}{V_s}\right] \cdot \frac{x'}{V_s} - \frac{k_z}{k_y} \cdot z' + \begin{cases} 0 \\ \frac{A}{k_y} \cdot cos(w \cdot \omega_0 \cdot t' - \pi) \end{cases}$$

$$\frac{1}{\omega_0} \cdot \frac{dz'}{dt'} = \frac{k_x}{k_z} \cdot x'$$

These equations can easily get implemented as blocks or as an electronic circuit.

Calculating back from per-unit-parameters:

$$\mu = 0.2, w = 1.15, A = [0..1]$$

and some assumptions:

$$C = \frac{100}{2\pi}\ \mu F, \omega_0 = 2\pi \cdot 1000\ \frac{rad}{s}, I_0 = 0.5\ A$$

we obtain physical parameters:

$$L = \frac{1}{{\omega_0}^2 \cdot C} = \frac{10}{2\pi}\ mH$$

$$R_0 = \mu \cdot \sqrt{\frac{L}{C}} = \frac{\mu}{\omega_0 \cdot C} = 2\ \Omega$$

$$\hat{V} = A \cdot \frac{I_0}{\omega_0 \cdot C} = [0 \cdots 5]\ V$$

Investigating the nonlinear resistance of the triode:

$$v_R = -R_0 \cdot I_0 \cdot \left[\left(\frac{i}{I_0}\right) - \frac{1}{3} \cdot \left(\frac{i}{I_0}\right)^3\right] = -R_0 \cdot i \cdot \left[1 - \frac{1}{3} \cdot \left(\frac{i}{I_0}\right)^2\right]$$

$$\frac{v_R}{i} = -R_0 \cdot \left[1 - \frac{1}{3} \cdot \left(\frac{i}{I_0}\right)^2\right]$$

$$\frac{dv_R}{di} = -R_0 \cdot \left[1 - \left(\frac{i}{I_0}\right)^2\right]$$

$$x = \frac{i}{I_0}$$

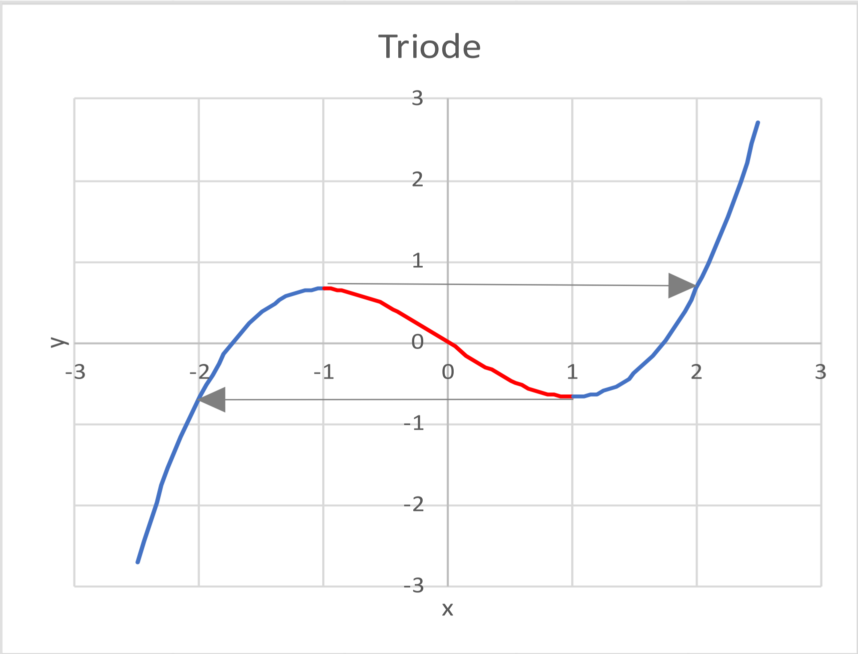
$$y = \frac{v_R}{R_0 \cdot I_0}$$

$$y = -\left(x - \frac{x^3}{3}\right) = -x \cdot \left(1 - \frac{x^2}{3}\right)$$

$$\frac{y}{x} = -\left(1 - \frac{x^2}{3}\right)$$

$$\frac{dy}{dx} = -(1 - x^2)$$

Characteristic of the triode $y(x)$:



Zero crossings $y = 0$: $x = \{-\sqrt{3}, 0, +\sqrt{3}\}$ with slopes $\frac{dy}{dx} = \{+2, -1, +2\}$

Extrema: $x = \{-1, +1\}$ with $x = \left\{+\frac{2}{3}, -\frac{2}{3}\right\}$

Inflection point: $[x, y] = [0, 0]$

If current x is prescribed, voltage y can be unambiguously determined.

If y is prescribed, in the range $-2 \le x \le +2$ i.e. $-\frac{2}{3} \le y \le +\frac{2}{3}$ x has 2 or 3 possible solutions.

For this application, this restriction has no influence.

Shifting the characteristic up and to the right, it looks like the $i(v)$ characteristic of a tunnel (Esaki) diode.

Inversion of the triode characteristic shows hysteretic behavior (split into 2 branches):

$$x \ge +1: y + \frac{2}{3} = \frac{(x-1)^3}{3} + (x-1)^2$$

$$x \le -1: y - \frac{2}{3} = \frac{(x+1)^3}{3} - (x+1)^2$$

Lorenz System

[Buscarino2014], [Lorenz1963]

Developed 1963 by Edward Lorenz to model atmospheric convection.
x is proportional to the rate of convection, y to the horizontal temperature variation and z to the vertical temperature variation. σ depicts the Prandtl number, ρ the Rayleigh number and β the physical dimensions.
The original parameters were: $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$

$$\frac{dx}{dt} = \sigma \cdot (y - x)$$
$$\frac{dy}{dt} = x \cdot (\rho - z) - y$$
$$\frac{dz}{dt} = x \cdot y - \beta \cdot z$$

$\beta = \frac{1}{3}$ leads to a periodic solution.

For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. This can be compared with calculating per-unit values by dividing by reference values:

$$x' = \frac{x}{k_x}$$
$$y' = \frac{y}{k_y}$$
$$z' = \frac{z}{k_z}$$
$$t' = \frac{t}{\tau}$$

We also have to take into account that the analog multiplier divides by V_S to avoid overflow of the output.
After that, none of the computing block should encounter an overflow.
This leads to the following set of equations:

$$\frac{1}{\tau} \cdot \frac{dx'}{dt'} = -\sigma \cdot x' + \sigma \cdot \frac{k_y}{k_x} \cdot y'$$
$$\frac{1}{\tau} \cdot \frac{dy'}{dt'} = \rho \cdot \frac{k_x}{k_y} \cdot x' - y' - \frac{k_x \cdot k_z \cdot V_S}{k_y} \cdot \frac{x' \cdot z'}{V_S}$$
$$\frac{1}{\tau} \cdot \frac{dz'}{dt'} = \frac{k_x \cdot k_y \cdot V_S}{k_z} \cdot \frac{x' \cdot y'}{V_S} - \beta \cdot z'$$

These equations can easily get implemented as blocks or as an electronic circuit.

Roessler System

[Roessler1976], [Roessler1979]

A simple system of 3 ordinary nonlinear differential equations to study chaos without physical background.

$$\frac{dx}{dt} = -y - z$$
$$\frac{dy}{dt} = x + a \cdot y$$
$$\frac{dz}{dt} = b + (x - c) \cdot z$$

$a = 0.2$, $b = 0.2$ and $c = 1$ give periodic results. Changing $c = 5.7$ reveals chaotic results.
For an implementation as an electronic circuit, the equations have to be scaled to keep the variables within the desired range. This can be compared with calculating per-unit values by dividing by reference values:

$$x' = \frac{x}{k_x}$$
$$y' = \frac{y}{k_y}$$
$$z' = \frac{z}{k_z}$$
$$t' = \frac{t}{\tau}$$

We also have to take into account that the analog multiplier divides by V_S to avoid overflow of the output. After that, none of the computing block should encounter an overflow.
This leads to the following set of equations:

$$\frac{1}{\tau} \cdot \frac{dx'}{dt} = -\frac{k_y}{k_x} \cdot y' - \frac{k_z}{k_x} \cdot z'$$
$$\frac{1}{\tau} \cdot \frac{dy'}{dt} = \frac{k_x}{k_y} \cdot x' + a \cdot y'$$
$$\frac{1}{\tau} \cdot \frac{dz'}{dt} = \frac{b}{k_z} + k_x \cdot V_S \cdot \frac{x' \cdot z'}{V_S} - c \cdot z'$$

These equations can easily get implemented as blocks or as an electronic circuit.

Chua's Circuit

[Berkeley], [Buscarino2014], [Chua1983], [Kennedy1993a], [Muthuswamy2009], [Zhong1985]

$$L \cdot \frac{di_L}{dt} = v_2 - R_L \cdot i_L$$

$$C_2 \cdot \frac{dv_2}{dt} = -i_L - \frac{v_2 - v_1}{R}$$

$$C_1 \cdot \frac{dv_1}{dt} = -i_{NL} + \frac{v_2 - v_1}{R}$$

$$-i_{NL}(v_1) = \begin{cases} -\infty < v_1 < -V_e \rightarrow G_b \cdot (v_1 + V_e) - G_a \cdot V_e \\ -V_e < v_1 < +V_e \rightarrow G_a \cdot v_1 \\ +V_e < v_1 < +\infty \rightarrow G_b \cdot (v_1 - V_e) + G_a \cdot V_e \end{cases}$$

$$-\frac{i_{NL}}{v_1} = \begin{cases} -\infty < v_1 < -V_e \rightarrow G_b - (G_a - G_b) \cdot \dfrac{V_e}{v_1} \\ -V_e < v_1 < +V_e \rightarrow G_a \\ +V_e < v_1 < +\infty \rightarrow G_b + (G_a - G_b) \cdot \dfrac{V_e}{v_1} \end{cases}$$

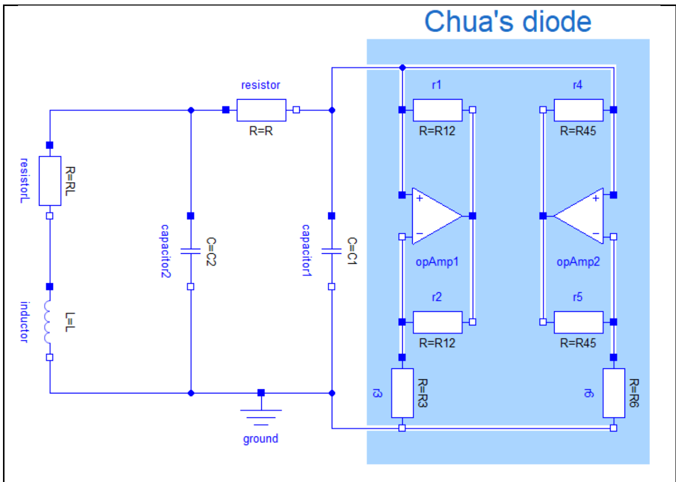
$$-\frac{di_{NL}}{dv_1} = \begin{cases} -\infty < v_1 < -V_e \rightarrow G_b \\ -V_e < v_1 < +V_e \rightarrow G_a \\ +V_e < v_1 < +\infty \rightarrow G_b \end{cases}$$

$$\tau_L \cdot \dot{v}_{RL} = v_2 - v_{RL}$$

$$\tau_2 \cdot \dot{v}_2 = +v_1 - v_2 - \frac{R}{R_L} \cdot v_{RL}$$

$$\tau_1 \cdot \dot{v}_1 = -v_1 + v_2 + R \cdot g \cdot v_1$$

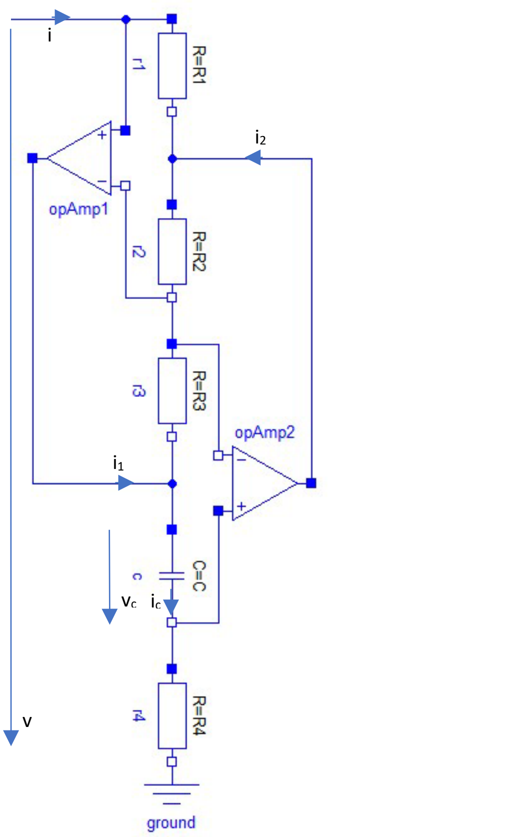
$$g(v_1) = \begin{cases} |v_1| > V_e \rightarrow G_b + (G_a - G_b) \cdot \dfrac{V_e}{|v_1|} \\ |v_1| < V_e \rightarrow G_a \end{cases}$$




```
parameter SI.Resistance R=1.9e3 "Resistor";
parameter SI.Inductance L=18e-3 "Inductor";
parameter SI.Resistance RL=14 "Resistance of Inductor";
parameter SI.Capacitance C1=10.e-9 "Capacitor 1";
parameter SI.Capacitance C2=100e-9 "Capacitor 2";
//parameter of Chua's diode
parameter Real k0=15000.0 "No-load amplification ";
parameter SI.Voltage Vs=9 "Supply voltage of opAmps";
parameter SI.Resistance R12=220 "R1 and R2";
parameter SI.Resistance R3=2200 "R3";
parameter SI.Resistance R45=22e3 "R4 and R5";
parameter SI.Resistance R6=3300 "R6";
```

This implementation of Chua's Diode with opAmps combines two NICs – see NIC (negative impedance converter).

Chua's Circuit: Inductor Replacement



OpAmp input currents neglectible
OpAmp differential input voltage neglectible

$$R_1 \cdot i + R_2 \cdot (i + i_2) = 0$$
$$R_3 \cdot (i + i_2) + v_c = 0$$
$$i + i_2 + i_1 = i_c = C \cdot \frac{dv_c}{dt}$$

$$i + i_2 = -\frac{R_1}{R_2} \cdot i$$
$$v_c = \frac{R_1 \cdot R_3}{R_2} \cdot i$$
$$i_c = C \cdot \frac{R_1 \cdot R_3}{R_2} \cdot \frac{di}{dt}$$
$$v = R_4 \cdot i_c = C \cdot \frac{R_1 \cdot R_3 \cdot R_4}{R_2} \cdot \frac{di}{dt}$$

This TwoPin is *not* a OnePort $i_c \neq i$ and $i_1 + i_2 \neq 0$!
The ground at the bottom is necessary.

$$R_1 = 100\ \Omega$$
$$R_2 = 1\ k\Omega$$
$$R_3 = 1\ k\Omega$$
$$R_4 = 1{,}8\ k\Omega$$
$$C = 100\ nF$$
$$L = C \cdot \frac{R_1 \cdot R_3 \cdot R_4}{R_2} = 18\ mH$$

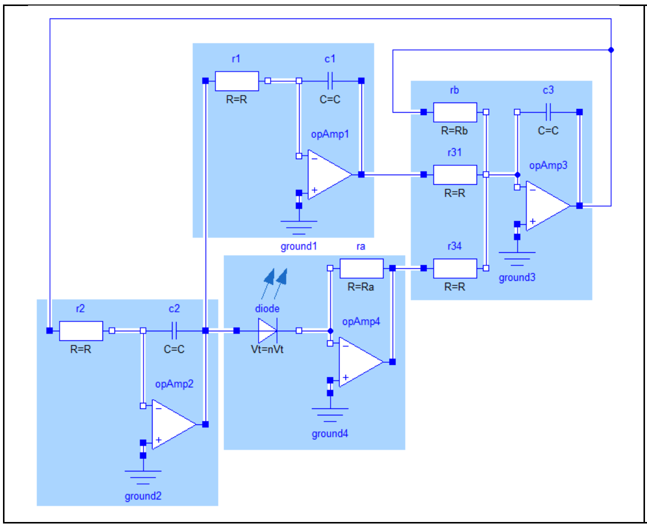
Chaotic Diode Circuit

[Pham2016]

$$C \cdot \frac{dv_1}{dt} = -\frac{v_2}{R}$$
$$C \cdot \frac{dv_2}{dt} = -\frac{v_3}{R}$$
$$C \cdot \frac{dv_3}{dt} = -\frac{v_1}{R} - \frac{v_3}{R_b} - \frac{v_4}{R}$$
$$\frac{v_4}{R_a} = -I_{ds} \cdot \left(e^{\frac{v_2}{nV_t}} - 1\right)$$

$$\tau = R \cdot C$$
$$\tau \cdot \frac{dv_1}{dt} = -v_2$$
$$\tau \cdot \frac{dv_2}{dt} = -v_3$$
$$\tau \cdot \frac{dv_3}{dt} = -v_1 - \frac{R}{R_b} \cdot v_3 + R_a \cdot I_{ds} \cdot \left(e^{\frac{v_2}{nV_t}} - 1\right)$$

$$a = \frac{R_a \cdot I_{ds}}{nV_t}$$
$$b = \frac{R}{R_b}$$
$$\tau \cdot \dot{x}_1 = -x_2$$
$$\tau \cdot \dot{x}_2 = -x_3$$
$$\tau \cdot \dot{x}_3 = -x_1 + a \cdot (e^{x_2} - 1) - b \cdot x_3$$




```
parameter SI.Current Ids=1e-9 "Diode: saturation current";
parameter SI.Voltage nVt=2*26e-3
" n*voltage equ. of temperature";
//time constant and dimensionless parameters
parameter SI.Time Tau=0.1e-3 "Time constant";
parameter Real a=1/5200 "Parameter a = Ra*Ids/nVt";
parameter Real b=1.5 "Parameter b = R/Rb";
//configuration of the opAmps
parameter SI.Capacitance C=10e-9
"Capacitance of {c1, c2, c3}";
parameter SI.Resistance R=Tau/C "Resistance of {r1,r2,r3,r4}";
parameter SI.Resistance Ra=a*nVt/Ids "Resistance of ra";
parameter SI.Resistance Rb=R/b "Resistance of rb";
```

Chaotic Oscillator

[Tamasevicius2005], [Tamasevicius2007]

$$i_L = C \cdot \frac{dv_C}{dt}$$

$$L \cdot \frac{di_L}{dt} = \left(k - 1 - \frac{R_L}{R}\right) \cdot R \cdot i_L - v_C - v_{C^*}$$

$$k = 1 + \frac{R_2}{R_1}$$

$$C^* \cdot \frac{dv_{C^*}}{dt} = I_0 + i_L - I_{DS} \cdot \left(e^{\frac{v_{C^*}}{nV_t}} - 1\right)$$

$$I_0 \approx \frac{V_b}{R_0}$$

$$\tau = \sqrt{L \cdot C}$$

$$Z = \sqrt{\frac{L}{C}}$$

$$a = \left(k - 1 - \frac{R_L}{R}\right) \cdot \frac{R}{Z}$$

$$b = \frac{Z \cdot I_0}{nV_t}$$

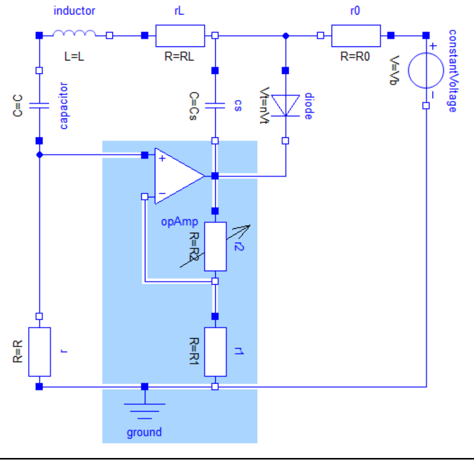
$$c = \frac{Z \cdot I_{DS}}{nV_t}$$

$$e = \frac{C^*}{C}$$

$$\tau \cdot \frac{\dot{v}_C}{nV_t} = \frac{Z \cdot i_L}{nV_t}$$

$$\tau \cdot \frac{Z \cdot \dot{i}_L}{nV_t} = \left(k - 1 - \frac{R_L}{R}\right) \cdot \frac{R}{Z} \cdot \frac{Z \cdot i_L}{nV_t} - \frac{v_C}{nV_t} - \frac{v_{C^*}}{nV_t}$$

$$\tau \cdot e \cdot \frac{\dot{v}_{C^*}}{nV_t} = \frac{Z \cdot I_0}{nV_t} + \frac{Z \cdot i_L}{nV_t} - \frac{Z \cdot I_{DS}}{nV_t} \cdot \left(e^{\frac{v_{C^*}}{nV_t}} - 1\right)$$




```
//time constant and dimensionless parameters
parameter SI.Current I0=Vb/R0 "Estimated supply current";
parameter SI.Time Tau=sqrt(L*C) "Time constant";
parameter SI.Impedance Z=sqrt(L/C) "Characteristic impedance (rho)";
parameter Real k=Z/R*a + 1 + RL/R "Amplification";
parameter Real a=0.95;
parameter Real b=Z*I0/nVt;
parameter Real c=Z*Ids/nVt;
parameter Real e=Cs/C "epsilon";
//components
parameter SI.Resistance R =1.0e3 "Resistor r";
parameter SI.Resistance R1=10.e3 "Resistor r1";
parameter SI.Resistance R2=R1*(k - 1) "Resistor r2 (tunable)";
parameter SI.Resistance R0=20.e3 "Resistor r0";
parameter SI.Voltage Vb=20 "Supply voltage";
parameter SI.Inductance L=100e-3 "Inductor";
parameter SI.Resistance RL=50 "Resistance of Inductor";
parameter SI.Capacitance C =100e-9 "Capacitor c";
parameter SI.Capacitance Cs=15.e-9 "Capacitor c*";
//diode
parameter SI.Current Ids=1e-9 "Diode: saturation current";
parameter SI.Voltage nVt=2*26e-3
"Diode: n * voltage equivalent of temperature";
```

Colpitts Oscillator

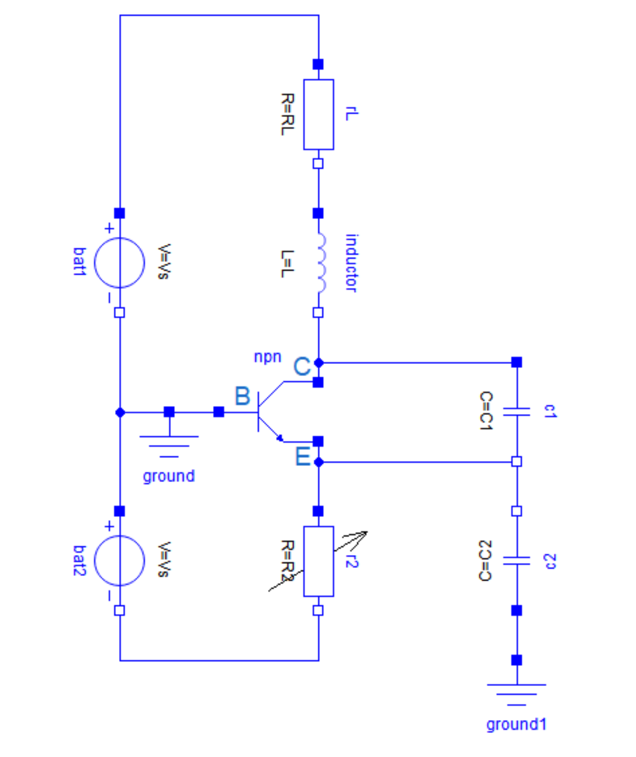
[Buscarino2014], [Kennedy1994]

$$C_1 \cdot \frac{dv_1}{dt} = i_L - \beta \cdot i_B$$

$$C_2 \cdot \frac{dv_2}{dt} = -\frac{V_{s-} + v_2}{R_2} - i_L - i_B$$

$$L \cdot \frac{di_L}{dt} = V_{s+} - v_1 + v_2 - R_L \cdot i_L$$

$$i_B = \begin{cases} v_2 = v_{BE} \leq V_{th} \rightarrow & 0 \\ v_2 = v_{BE} > V_{th} \rightarrow & \dfrac{v_2 - V_{th}}{R_{on}} \end{cases}$$




```
parameter SI.Resistance RL=35. "Resistance of L";
parameter SI.Inductance L=98.5e-6 "Inductor";
parameter SI.Resistance R2=1000 "Resistor 2";
parameter SI.Capacitance C1=54.e-9 "Capacitor 1";
parameter SI.Capacitance C2=54.e-9 "Capacitor 2";
parameter SI.Voltage Vs=5 "Source Voltage”;

parameter SI.Voltage Vth=0.75 “Transistor threshold voltage”;
parameter SI.Resistance Ron=100
  “Small-signal on-resistance of base-emitter junction”;
parameter Real beta=200 “Transistor forward current gain”;
```

Shinriki Oscillator

[Lueck1995], [Shinriki1981]

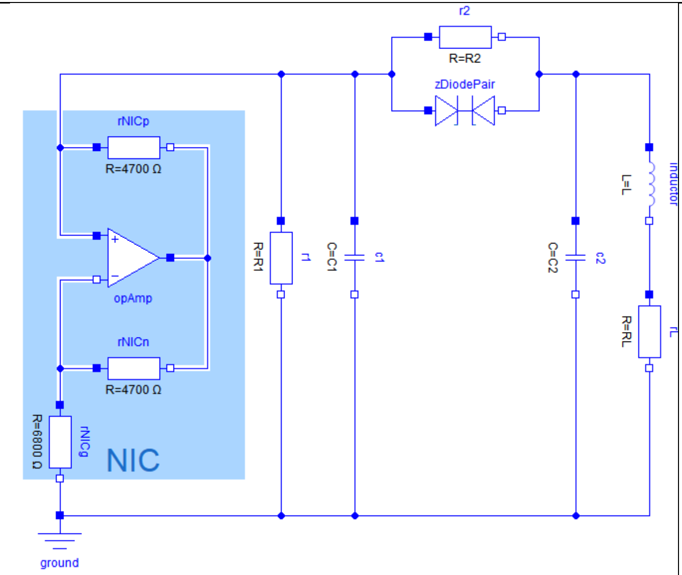
$$i_z = \begin{cases} |v_z| < V_{bt} & 0 \\ |v_z| \geq V_{bt} & sign(v_z) \cdot [a \cdot (|v_z| - V_{bt}) + b \cdot (|v_z| - V_{bt})^3 + c \cdot (|v_z| - V_{bt})^5] \end{cases}$$

$$C_1 \cdot \frac{dv_1}{dt} = -i_{NIC} - \frac{v_1}{R_1} - i_z$$

$$C_2 \cdot \frac{dv_2}{dt} = i_z - i_L$$

$$v_2 = L \cdot \frac{di_L}{dt} + R_L \cdot i_L$$

$$g_{NIC} = \frac{di_{NIC}}{dv_{NIC}} = \begin{cases} |v_{NIC}| > V_{Lim} & g_+ \\ |v_{NIC}| \leq V_{Lim} & g_- \end{cases}$$




```
parameter SI.Inductance L=320e-3 "Inductor";
parameter SI.Resistance RL=100. "Resistor of L";
parameter SI.Resistance R1=60e3 "Resistor 1";
parameter SI.Resistance R2=20e3 "Resistor 2";
parameter SI.Capacitance C1=10.e-9 "Capacitor 1";
parameter SI.Capacitance C2=100e-9 "Capacitor 2";
```

$$V_{bt} = 3.3\ V$$

$$a = 1{,}0862\ \frac{mA}{V}$$

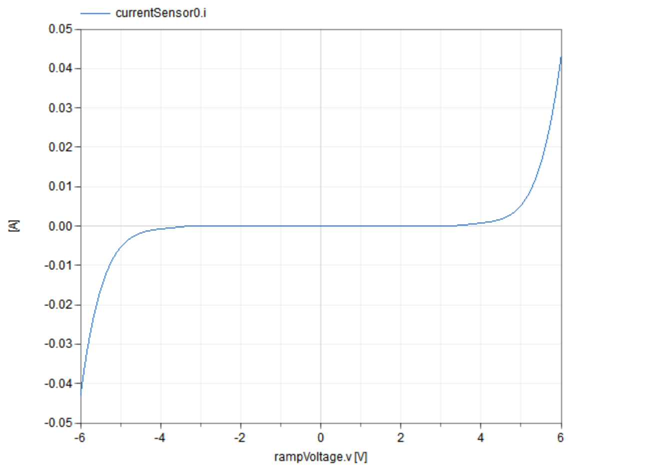
$$b = -0{,}1615\ \frac{mA}{V^3}$$

$$c = 0{,}3021\ \frac{mA}{V^5}$$

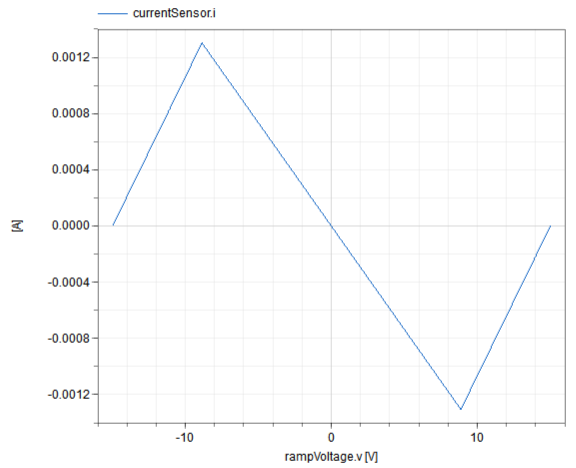
$$V_{Lim} = V_s \cdot \frac{6800}{4700 + 6800}$$

$$g_+ = +\frac{1000}{4700}\ mS$$

$$g_- = -\frac{1000}{6800}\ mS$$



Characteristic of the Zener diode pair



Characteristic of the NIC (negative impedance converter)

Investigation of the Zener diode pair approximation:

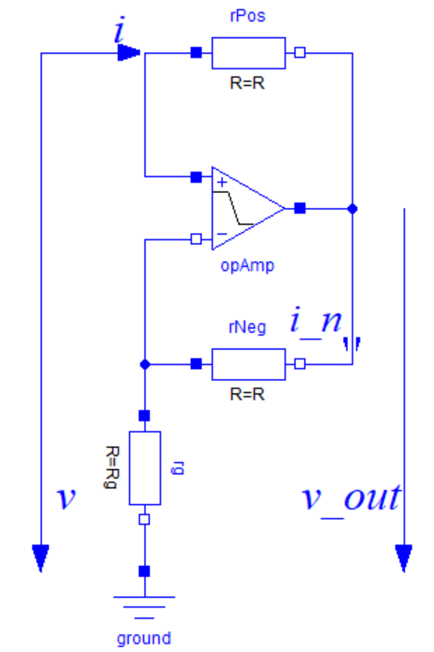
$$i_z = \begin{cases} |v_z| < V_{bt} & 0 \\ |v_z| \geq V_{bt} & sign(v_z) \cdot [a \cdot (|v_z| - V_{bt}) + b \cdot (|v_z| - V_{bt})^3 + c \cdot (|v_z| - V_{bt})^5] \end{cases}$$

$$\frac{di_z}{dv_z} = \begin{cases} |v_z| < V_{bt} & 0 \\ |v_z| \geq V_{bt} & sign(v_z) \cdot [a + 3b \cdot (|v_z| - V_{bt})^2 + 5c \cdot (|v_z| - V_{bt})^4] \end{cases}$$

The first derivative $\frac{di_z}{dv_z}$ is not continuous at $|v_z| = V_{bt}$.

If voltage v is prescribed, the current i can be unambiguously determined.
If current i is prescribed, between $-V_{bt}$ and $+V_{bt}$ there is a manifold of solutions for the voltage v.
For this application, this restriction has no influence.
This restriction could be solved by adaption the approximation, i.e. exchange the horizontal line in the range $-V_{bt} < v_z < +V_{bt}$ against a characteristic with small constant positive slope and adapt the polynomial approximation to achieve a one times continuously differentiable characteristic.

NIC (negative impedance converter)



As long as the opAmp operates in the linear region:

$$i_n = \frac{v_{out} - v}{R} = \frac{v}{R_g} \rightarrow v_{out} = v \cdot \frac{R + R_g}{R_g}$$

$$i = \frac{v - v_{out}}{R} = -\frac{v}{R_g}$$

$$g_- = \frac{1}{R_g}$$

When the opAmp's output saturates:

$$V_{Lim} = V_s \cdot \frac{R_g}{R + R_g}$$

$$v \geq +V_{Lim}: i = \frac{v - V_{Lim}}{R}$$

$$g_+ = \frac{1}{R}$$

If voltage v is prescribed, the current i can be unambiguously determined.
If current i is prescribed, in the range between the zero crossings the voltage v has 3 possible solutions.
For this application, this restriction has no influence.

Jerk Circuit

[Buscarino2014], [Sprott2011]

The name of the system stems from the third derivative of x, which – in a mechanical system – is the derivative of acceleration called jerk. The Jerk equation has been investigated in different versions.

$$\dddot{x} + G(\ddot{x}, \dot{x}, x) = 0$$

The version implemented here uses a diode as described in the mentioned publications:

$$G(\ddot{x}, \dot{x}, x) = A \cdot \ddot{x} + f(\dot{x}) + x$$
$$\dddot{x} + A \cdot \ddot{x} + f(\dot{x}) + x = 0$$

$f(\dot{x})$ is modeled using the Shockley equation of a diode:

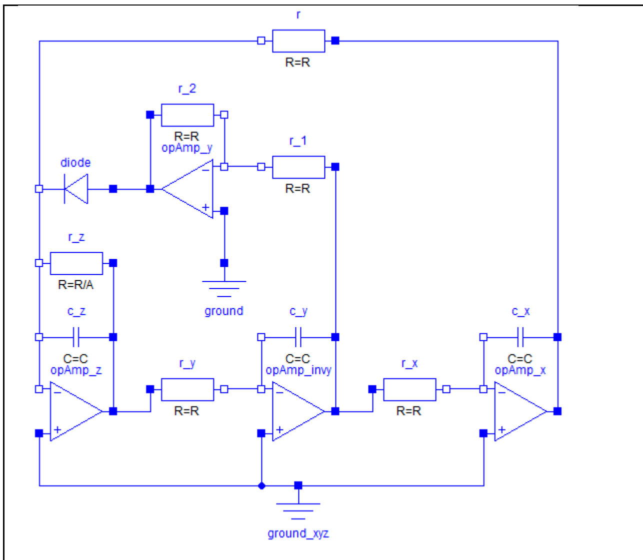
$$f(\dot{x}) = R \cdot I_S \cdot \left(e^{\frac{\dot{x}}{nV_t}} - 1\right)$$

This leads to a system of 3 ordinary differential equations with one nonlinearity:

$$\dot{x} = y$$
$$\ddot{x} = \dot{y} = z$$
$$\dddot{x} = \ddot{y} = \dot{z}$$
$$\dot{z} = -A \cdot z - x - f(y)$$

The values stay pretty inside a practicable range for a normal voltage supply.

Using 4 operational amplifiers and an acceleration factor of 1000, the circuit can be implemented as follows. The parameter A influences only the feedback resistor at opAMp_z:




```
parameter Real A=0.3 "Parameter to be varied";
parameter SI.Resistance R=1e3 "Resistance";
parameter SI.Capacitance C=1e-6 "Capacitance";
parameter SI.Current Ids=1e-12 "Sat.current";
parameter SI.Voltage nVt=26e-3 " voltage equ.";
```

A=0.3 for periodic results

A=1.0 for chaotic results

Rikitake System

[Rikitake1958], [Rikitake1973]

The system proposed by Rikitake has been used to explain irregular reversals of the Earth's magnetic field.
2 identical magnetically coupled disc dynamos ($\tau \cdot \omega$ covers the losses $R \cdot i^2$):

$$L \cdot \frac{di_1}{dt} + R \cdot i_1 = M \cdot i_2 \cdot \omega_1$$
$$J \cdot \frac{d\omega_1}{dt} = \tau - M \cdot i_2 \cdot i_1$$
$$L \cdot \frac{di_1}{dt} + R \cdot i_2 = M \cdot i_1 \cdot \omega_2$$
$$J \cdot \frac{d\omega_2}{dt} = \tau - M \cdot i_1 \cdot i_2$$

$\phi_{12} = M \cdot i_2$ is the magnetic flux in machine 1 excited by current i_2,
$\phi_{21} = M \cdot i_1$ is the magnetic flux in machine 2 excited by current i_1.
The circuit could be implemented using two series excited DC machines
with the armature current of the other machine as excitation current.
The equations of motion have identical right hand sides:

$$\frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} \rightarrow \omega_1 - \omega_2 = \Delta\omega = const.$$

Mechanical and electrical time constant:

$$T_m = \frac{J}{\tau} \cdot \frac{R}{M}$$
$$T_e = \frac{L}{R}$$
$$\mu = \sqrt{\frac{T_m}{T_e}} = \sqrt{\frac{J}{\tau} \cdot \frac{R^2}{L \cdot M}}$$

used to scale the variables:

$$i_{1,2} = x_{1,2} \cdot \sqrt{\frac{\tau}{M}}$$
$$\omega_1 = z \cdot \sqrt{\frac{\tau}{J} \cdot \frac{L}{M}} \rightarrow \omega_2 = (z - \Delta) \cdot \sqrt{\frac{\tau}{J} \cdot \frac{L}{M}}$$
$$t = t' \cdot \sqrt{T_m \cdot T_e} = t' \cdot \sqrt{\frac{J}{\tau} \cdot \frac{L}{M}}$$

lead to the scaled equations:

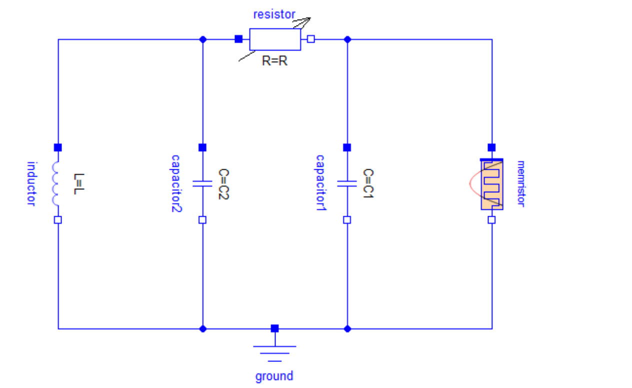
$$\frac{dx_1}{dt'} = -\mu \cdot x_1 + x_2 \cdot z$$
$$\frac{dx_2}{dt'} = -\mu \cdot x_2 + x_1 \cdot (z - \Delta)$$
$$\frac{dz}{dt'} = 1 - x_1 \cdot x_2$$

The states stay within a range that needs no scaling when implemented with an opAmp-circuit.

Chua’s Circuit with Memristor

[Muthuswamy2010a]

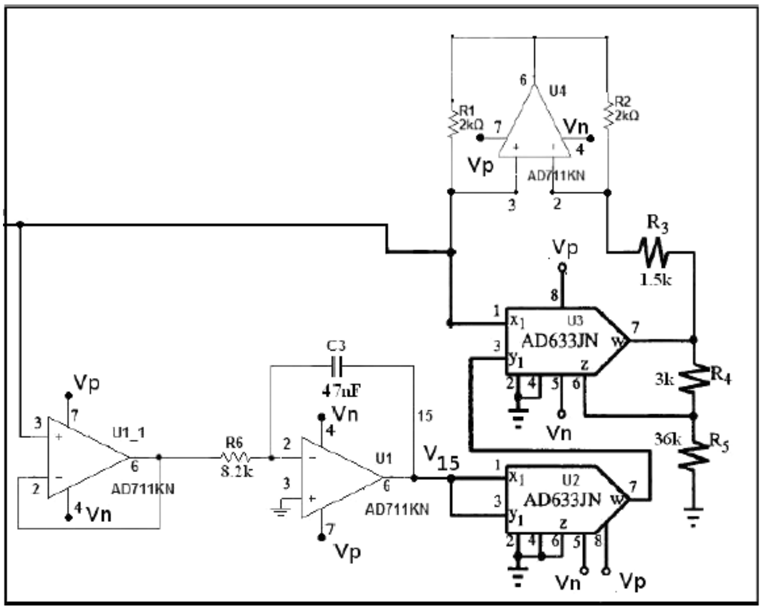
Muthuswamy suggested 2010 a special memristor function to replace Chua’s diode:



$$v_2 = L \cdot \frac{di_L}{dt}$$
$$C_2 \cdot \frac{dv_2}{dt} + i_L + \frac{v_2 - v_1}{R} = 0$$
$$C_1 \cdot \frac{dv_1}{dt} + i_M - \frac{v_2 - v_1}{R} = 0$$
$$i_M = G_{Ref} \cdot (\alpha + 3 \cdot \beta \cdot \xi^2) \cdot v_1$$
$$\phi_{Ref} \cdot \frac{d\xi}{dt} = v_1$$

ϕ_{Ref} is choosen to scale the system.

Replacement of the memristor according to Fig. 6 of [Muthuswamy2010a]:



Note that the scaling factor has been adapted (adapting R6 and/or C3):

$$\phi_{Ref} = \frac{8200 \cdot 47 \cdot 10^{-9}}{10}$$

The division by 10 of the analog multipliers is taken into account with the resistors R3, R4, R5.

Note that the summing input z of the analog multiplier U3 is used.

This way the circuit can be implemented by standard electronic components.

A voltage follower U1_1 is used as a buffer.

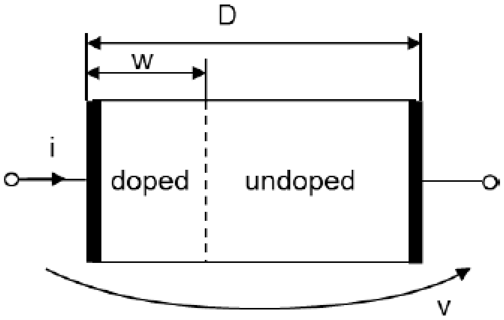
Note that this circuit implements a different relationship between flux and voltage:

$$\frac{d\phi}{dt} = -v_1$$

Memristor models

[Biolek2009], [Biolek2015], [Chua1971], [Chua1977], [Itoh2008], [Joglekar2009], [Majetta2012], [Maurer2014], [Muthuswamy2010], [Muthuswamy2010a], [Oguz2018], [Strukov2008], [Yakopcic2011]

Charge-controlled memristor State = charge: $i = \frac{dq}{dt}$ which remembers the history of current.	$\phi = f_\phi(q)$ $v = \frac{d\phi}{dt} = \frac{df_\phi}{dq} \cdot \frac{dq}{dt} = R_{mem} \cdot i$ $i = \frac{dq}{dt} = f_q(q, i)$
Flux-controlled memristor State = magnetic flux: $v = \frac{d\phi}{dt}$ which remembers the history of voltage.	$q = f_q(\phi)$ $i = \frac{dq}{dt} = \frac{df_q}{d\phi} \cdot \frac{d\phi}{dt} = G_{mem} \cdot v$ $v = \frac{d\phi}{dt} = f_\phi(\phi, v)$

[Majetta2012] Charge-controlled memristor (HP) drift model: $x = \frac{w}{D} = \frac{q}{q_{Ref}}$  	$R_{mem} = R_{on} \cdot x + R_{off} \cdot (1 - x)$ $\frac{dx}{dt} = \frac{\mu_v \cdot R_{on}}{D^2} \cdot i \cdot fw(x)$ μ_v is the dopant mobility: $[\mu_v] = \frac{m^2}{V \cdot s}$ $fw(x)$ is a window-function. [Joglekar2009] $fw(x) = 1 - (2x - 1)^{2p}$ [Biolek2009] $fw(x) = 1 - \left(x - \begin{cases} 1 & -i \geq 0 \\ 0 & -i < 0 \end{cases}\right)^{2p}$

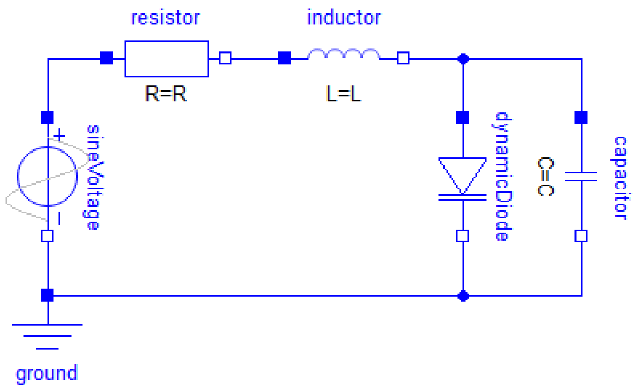
[Muthuswamy2010] Charge-controlled memristor $v = \beta \cdot (x^2 - 1) \cdot \mathrm{i}$ $\frac{dx}{dt} = i - \alpha \cdot x - x \cdot i$	$R_{mem} = R_0 \cdot (\xi^2 - 1)$ $\xi = \frac{q}{q_{Ref}} = \sqrt{1 + \frac{R_{mem}}{R_0}}$ $T \cdot \frac{d\xi}{dt} = \frac{i}{I_0} - \xi - \xi \cdot \frac{i}{I_0}$

Appendix B.3 of [Muthuswamy2010] demonstrates the implementation of this special memristor using operational amplifiers and analog multipliers.

[Muthuswamy2010a] Flux-controlled memristor $\frac{dq}{dt} = i = (\alpha + 3 \cdot \beta \cdot \phi^2) \cdot v$ $q = \alpha \cdot \phi + \beta \cdot \phi^3$ $\frac{d\phi}{dt} = v$	$G_{mem} = G_{Ref} \cdot (\alpha + 3 \cdot \beta \cdot \xi^2)$ $\xi = \frac{\phi}{\phi_{Ref}} = \sqrt{\frac{G_{mem}}{G_{Ref} \cdot 3 \cdot \beta} - \frac{\alpha}{3 \cdot \beta}}$ $\phi_{Ref} \cdot \frac{d\xi}{dt} = \frac{d\phi}{dt} = v$

Fig. 6 of [Muthuswamy2010a] demonstrates the implementation of this special memristor using operational amplifiers and analog multipliers.

RLD - Resonator



[Hellen2024]
The dynamic diode model in parallel to the Shockley-equation junction and diffusion capacitance.
I_{ds} ... Reverse saturation current
nV_t ... $n \times$ voltage equivalent of temperature
[Tietze2019]
Note: High current injection and break through are not taken into account.
Natural frequency:

$$2\pi \cdot f_0 = \frac{1}{\sqrt{L \cdot \left(C + C_{J(v=0)} + C_{D(v=0)}\right)}}$$

$$\hat{V} \cdot sin(2\pi \cdot f \cdot t) = R \cdot i + L \cdot \frac{di}{dt} + v_C$$

$$i = (i_D + i_C) + C \cdot \frac{dv_C}{dt}$$

$$i_D = I_{ds} \cdot \left(e^{\frac{v_C}{nV_t}} - 1\right)$$

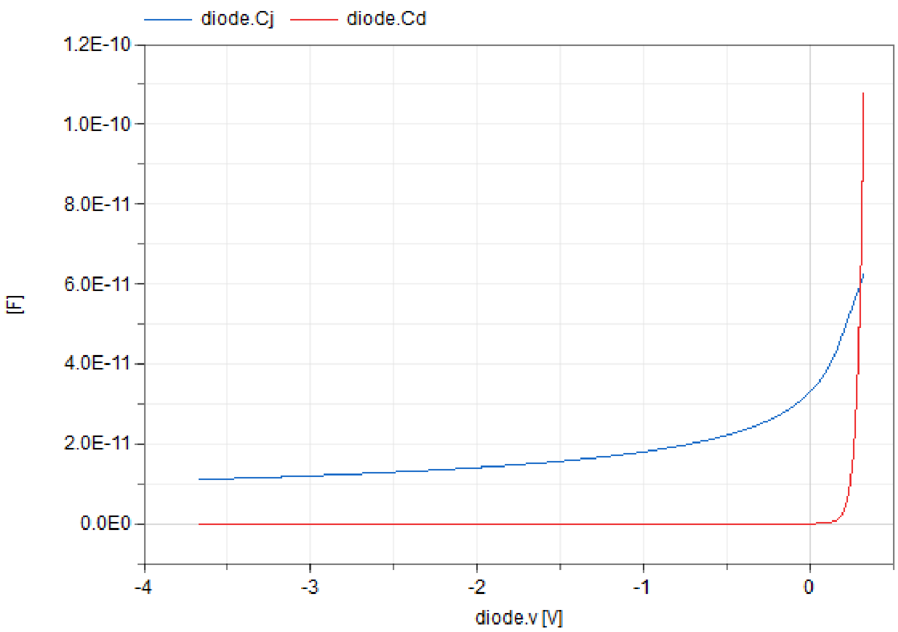
$$i_C = \frac{dq}{dt}$$

$$q = \left(C_J + C_D\right) \cdot v_C$$

$$C_J = \begin{cases} v_C \leq f_c \cdot V_0 & \dfrac{C_0}{\left(1 - \frac{v_C}{V_0}\right)^m} \\ v_C > f_c \cdot V_0 & C_0 \cdot \dfrac{1 - f_c \cdot (1+m) + m \cdot \frac{v_C}{V_0}}{(1 - f_c)^{1+m}} \end{cases}$$

$$C_D = \frac{\tau_T \cdot I_{ds}}{nV_t} \cdot \left(e^{\frac{v_C}{nV_t}} - 1\right)$$

Parameters of the junction capacitance are C_0, V_0, m and f_c. τ_T is the transition time.



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